5
Local Crushing

In this section we show how to perturb the vector field inside a tubular neighborhood of an orbit segment in order to obtain the crushing property defined in Lemma 2.5.1, relatively to the volume of the neighborhood. The name "Fettuccine" given for the main lemma of the section is due to the fact that crushable sets must have a specific geometry which resembles the fettuccine's shape. We now state this lemma (Lemma 5.0.22), which will be proved only in §5.4, after we present in the following subsections the necessary ingredients of the proof.

Lemma 5.0.22 (Fettuccine's Lemma) *Let $X \in \mathfrak{X}^3(M)$ and let Σ be a cross section. Then for all $\epsilon > 0$ and $0 < \delta < 1$ there exists $t_0 > 0$ such that for all $T_0 > 3t_0$ there exists $\kappa > 0$ such that for all $T \in (3t_0, T_0)$ and for all non-periodic point $p \in \Sigma$, there exists $\rho > 0$ such that given a κ-rectangle R centered in p with $diam(R) < \rho$, there exists $\tilde{X} \in \mathfrak{X}^1(M)$ with $\|\tilde{X} - X\|_{C^1} < \epsilon$, $\tilde{X} = X$ outside U, where*

$$U = \bigcup_{t\in[0,T]} \varphi^t(R),$$

and there exists $V \subset U$ such that if

$$U^- = \bigcup_{t\in[0,T-t_0]} \varphi^t(R) \quad \text{and} \quad U^+ = \bigcup_{t\in[t_0,T]} \varphi^t(R),$$

then

1. $V \subset U^-$;
2. $\dfrac{m(V)}{m(U^-)} > 1 - \delta$;
3. $\varphi^t_{\tilde{X}}(\overline{V}) \subset U \qquad \forall t \in [0, t_0]$;
4. $\dfrac{m(\varphi^{t_0}_{\tilde{X}}(V) \triangle U^+)}{m(U^+)} < \delta$.

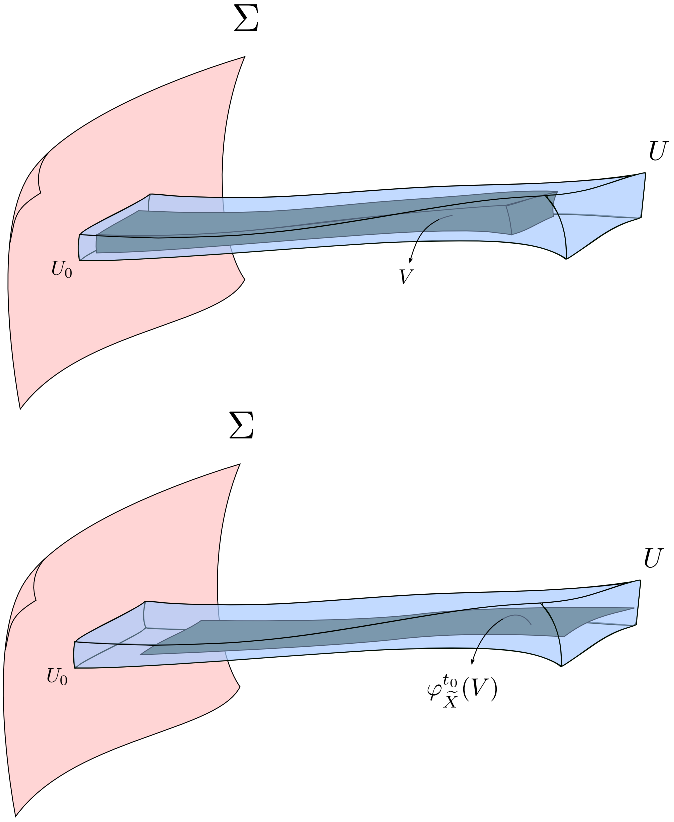

Figure 5.1: Schematic illustration of V being crushed.

The Tubular Chart Theorem (Theorem 4.0.15) set us in a very useful geometrical structure. Pulling back a vector field by the tubular chart, we obtain a vector field in an open set of $\mathbb{R}^d$ with several properties that will be frequently used in next subsections. For sake of clarity we call a vector field with those properties a *model* vector field.

Definition 5.0.23 (Model Vector Field) *Let* $a, C > 0$ *and* $T > 2a$ *be arbitrary constants. Let* $X \in \mathfrak{X}^1(U)$ *be a vector field defined in an open neighborhood* U *of the line* $\{(t, 0, 0) \in \mathbb{R} \times \mathbb{R}^{d-2} \times \mathbb{R} : t \in [-a, T + a]\}$. *We say that* X *is a* (C,T,a) - model vector field *if*

- $X(s, 0, 0) = (1, 0, 0), \quad \forall s \in [-a, T + a],$
- *the plane* $\{(x, w, y) : y = 0\}$ *is* X*-invariant,*
- $$\left| \frac{d}{dt} \log \|P_s^t\| \right| \leq C, \quad \forall s, t \in [-a, T + a],$$

  where P_s^t *is the linear isomorphism induced by the linear Poincaré flow based on the segment of orbit* $\{(x, 0, 0) : x \in [s, t]\}$.

Depending on the context, some of the constants C, T and a may be omitted from the notation of model vector fields. Possibly for they are implicitly understood, possibly for being unnecessary in some computation.

5.1 Crushing-Time

Once we have constructed a chart which permits us to work with model vector fields, next step will be defining the crushing-perturbation of a general model vector field. The main idea is to add small vertical vectors to the original vector field, bringing trajectories closer to the invariant plane and, consequently, crushing volume in that direction (See Figure 5.2). Since these new vertical components need to be very small, in order to obtain a significant "vertical deviation" of the original trajectories, a long period of time must elapse. This amount of time will be called *crushing-time* and its precise definition is given by Lemma 5.1.2.

Let $\pi_d : \mathbb{R} \times \mathbb{R}^{d-2} \times \mathbb{R}$ the standard projection in the d-th coordinate.

Then $D\pi_d(x, w, y)X(x, w, y)$ is simply the d-th coordinate of the vector $X(x, w, y)$. Define $\alpha : \mathbb{R} \times \mathbb{R}^{d-2}$ by

$$\alpha(x, w) = D(\pi_d \circ X)(x, w, 0) \cdot e_d.$$

Since X is C^1 we can write

$$D\pi_d(x, w, y)X(x, w, y) = \alpha(x, w)y + r(x, w, y), \tag{5.1}$$

where $r : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a continuous function satisfying

$$\lim_{y \to 0} \frac{r(x, w, y)}{y} = 0. \tag{5.2}$$

Proposition 5.1.1 *For all $\epsilon' > 0$ and $T_2 > T_1 > 0$ there exist $\rho > 0$ and such that if* $\max\{\|w\|, |y|\} < \rho$ *and $x \in [0, T_1]$ then*

$$\|\varphi^t(x, w, y) - (x + t, 0, 0)\| < \epsilon',$$

for all $t \in [0, T_2 - T_1]$, where φ^t is the flow generated by the vector field X.

Proof: This is an immediate consequence of the continuity of the flow and the first property of model vector fields. ■

Lemma 5.1.2 *Let $\epsilon > 0$ and $0 < \delta < 1$ be given. Then for any*

$$T > t_0 \equiv 3 - \frac{2\log(\delta)}{\epsilon},$$

for all $0 < a < 1$ *and for all model vector field* $X \in \mathfrak{X}^1(U)$ *(with respect to* a *and* T*), there exists* $\rho > 0$ *such that if* $z = (x, w, y) \in U$ *satisfies*

$$\max\{\|w\|, |y|\} < \rho \quad \textit{and} \quad x \in [0, T - t_0]$$

then

$$\widetilde{X}(x, w, y) = X(x, w, y) + (0, 0, -\epsilon y)$$

would be such that

$$\frac{|\pi_d(\varphi_{\widetilde{X}}^{\tilde{\tau}(z)}(z))|}{|\pi_d(\varphi_X^{\tau(z)}(z))|} < \delta,$$

where $\tilde{\tau}, \tau : N_{(x,0,0)} \to N_{(x+t_0,0,0)}$ *are the hitting-time functions defined in Subsection (2.1) with respect to time* t_0 *and to the fields* $\widetilde{X}$ *and* X*, respectively.*

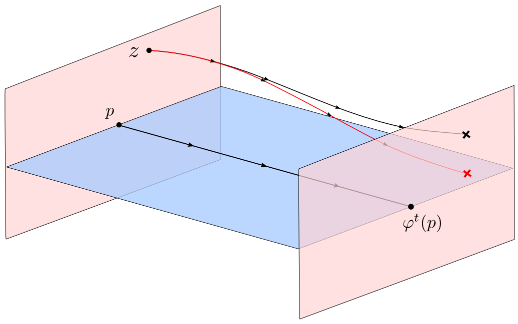

Figure 5.2: Crushing in the y-direction.

Proof: Notice that $\widetilde{X}$ is also a model vector field and thus we can obtain an equation similar to (5.1). Namely

$$D\pi_d(x, w, y) \cdot \widetilde{X}(x, w, y) = (\alpha(x, w) - \epsilon)y + r(x, w, y). \tag{5.3}$$

Suppose we have fixed a point $z = (x, w, y) \in \mathbb{R}^d$ sufficiently close to the x-axis (we will see later what this means). Let $\varphi_X^t(z) = (x(t), w(t), y(t))$ and $\varphi_{\widetilde{X}}^t(z) = (x_\epsilon(t), w_\epsilon(t), y_\epsilon(t))$ be the future paths of z generated by X and $\widetilde{X}$, respectively.

Then we have

$$\begin{aligned} y_\epsilon'(t) &= [\pi_d(\varphi_{\widetilde{X}}^t(z))]' \\ &= D\pi_d(\varphi_{\widetilde{X}}^t(z))\widetilde{X}(\varphi_{\widetilde{X}}^t(z)) \\ &= (\alpha(x_\epsilon(t), w_\epsilon(t)) - \epsilon)y_\epsilon(t) + r(x_\epsilon(t), w_\epsilon(t), y_\epsilon(t)). \end{aligned} \tag{5.4}$$

Similar computations lead us to

$$y'(t) = \alpha(x(t), w(t))y(t) + r(x(t), w(t), y(t)). \tag{5.5}$$

Proposition (5.1.1) together with the continuity of function α permit us to take $\rho > 0$ sufficiently small such that if $\max\{\|w\|, |y|\} < \rho$ and $x \in [-a, T - t_0]$ then

$$|\alpha(x(t), w(t)) - a_x(t)| < \epsilon/8 \quad \text{and} \quad |\alpha(x_\epsilon(t), w_\epsilon(t)) - a_x(t)| < \epsilon/8,$$

for all $t \in [0, t_0]$, where $a_x(s) \equiv \alpha(x + s, 0)$. Notice that the choice of ρ depends on T but does not depend on x.

By equation (5.2) and by the compactness of V, we can reduce ρ, if necessary, to guarantee that

$$|\tilde{r}(x_\epsilon(t), w_\epsilon(t), y_\epsilon(t))| \leq \frac{\epsilon|y_\epsilon(t)|}{8} \quad \text{and} \quad |r(x(t), w(t), y(t))| \leq \frac{\epsilon|y(t)|}{8},$$

for all $t \in [0, t_0]$ and for all (x, w) in the compact domain.

From these estimates and from equations (5.4) and (5.5) we can deduce the following inequalities:

$$y'_\epsilon(t) \leq (a_x(t) - 7\epsilon/8)y_\epsilon(t) + r(x_\epsilon(t), w_\epsilon(t), y_\epsilon(t)) \leq (a_x(t) - 3\epsilon/4)y_\epsilon(t) \tag{5.6}$$

$$y'(t) \geq (a_x(t) - \epsilon/8)y(t) + r(x(t), w(t), y(t)) \geq (a_x(t) - \epsilon/4)y(t). \tag{5.7}$$

Besides that, we want to consider $\rho > 0$ small enough (as in Remark 2.1.6) to obtain

$$|\tau(z) - t_0| < \min\left\{\frac{a}{2}, \frac{\epsilon}{4M}\right\} \quad \text{and} \quad |\widetilde{\tau}(z) - t_0| < \min\left\{\frac{a}{2}, \frac{\epsilon}{4M}\right\},$$

where

$$M = \sup_{s \in [-a, T+a]} |\alpha(s, 0)|.$$

Once we have all these estimates, we can conclude that

$$\frac{y_\epsilon(\tilde{\tau}(z))}{y(\tau(z))} \leq \frac{y_\epsilon(0)\exp(-3\epsilon\tilde{\tau}(z)/4)\exp\left(\int_0^{\tilde{\tau}(z)} a_x(t)dt\right)}{y(0)\exp(-\epsilon\tau(z)/4)\exp\left(\int_0^{\tau(z)} a_x(t)dt\right)}$$

$$\leq \exp(-3\epsilon(t_0-1)/4+\epsilon(t_0+1)/4)\exp\left(|\tilde{\tau}(z)-\tau(z)|\sup_{t\in[0,t_0+1]}|a_x(t)|\right)$$

$$\leq \exp\left(\frac{(2-t_0)\epsilon}{2}\right)\exp\left(\frac{\epsilon}{2M}M\right)$$

$$= \exp(\log(\delta)) = \delta.$$

■

5.2 Sliced Tube

In this Subsection, besides introducing some useful new notation, we will state and prove Proposition 5.1.1, that will allow us to work with more convenient sets - the sliced tubes, instead of the standard tubular neighborhoods.

We say that $\mathcal{B} \subset \mathbb{R}^d$ is a *ball* if it is a convex compact set, symmetric about the origin.

Definition 5.2.1 (K-Ball) *Let $B(0,r)$ denote the Euclidean ball with radius r. Given $K > 1$ we say that a ball $\mathcal{B}$ is a K-ball if*

$$B(0,K^{-1}) \subset \mathcal{B} \subset B(0,K).$$

Remark 5.2.2 *Let $K > 1$ and let $\mathcal{B} \subset \mathbb{R}^d$ be a K-ball. Then there exists a norm $\|\cdot\|_{\mathcal{B}}$ such that*

1. $\mathcal{B} = \{v \in \mathbb{R}^d : \|v\|_{\mathcal{B}} \leq 1\}$;
2. $K^{-1}\|v\| \leq \|v\|_{\mathcal{B}} \leq K\|v\| \quad \forall v \in \mathbb{R}^d$.

Remark 5.2.3 *Notice that, since the cube $\mathcal{B} = [-1,1]^d$ is inscribed in a sphere with radius $\sqrt{d}/2$ and circumscribed on a sphere with radius $1/2$, the hypothesis of Remark 5.2.2 is satisfied for any $K > \max\{2, \sqrt{d}/2\}$ and the conclusion, in this case, holds with $\|\cdot\|_{\mathcal{B}}$ being the norm of the maximum.*

Remark 5.2.4 *Let $P^t = P^t_0$ be the family of linear isomorphisms induced by the linear Poincaré flow over the base-orbit of a (C, T, Δ)-model vector field. Notice that, by the sub-exponential growth of $\|P^t\|$ and Remark 5.2.3, we can conclude that, for all $t \in [0, T + \Delta]$, $P^t([-1, 1]^{d-2})$ is a K-ball of $\mathbb{R}^{d-2}$ with*

$$K = \max\left\{2, \frac{\sqrt{d-2}}{2}\right\} e^{C(T+\Delta)}.$$

Fix a model vector field $X \in \mathfrak{X}^1(U)$ and let $P^t_s : \mathbb{R}^{d-1} \to \mathbb{R}^{d-1}$ denote the linear isomorphism induced by the linear Poincaré flow on the segment $[s, t]$ of the orbit of zero (Definition 2.3.8). Define also $\beta^t_s \equiv |\langle P^t_s(e_d), e_d\rangle|$, where $e_d = (0, 0, \ldots, 0, 1)$. Given an interval $I = [a, b]$ on the real line, $\kappa \in (0, 1)$, $r > 0$ and a ball $\mathcal{B} \subset \mathbb{R}^{d-2}$ we define a sliced tube $\mathcal{S}_{\mathcal{B}}(\kappa, r, I)$ by

$$\mathcal{S}_{\mathcal{B}}(\kappa, r, I) \equiv \bigcup_{s\in I} (\{s\} \times P^s_a(r\mathcal{B} \times \{0\}) \times [-\kappa r\beta^s_a, \kappa r\beta^s_a]) \,.$$

We most frequently deal with a special type of sliced tube, for which the ball and the height of the slices depend on the left edge of I.

Definition 5.2.5 (Standard Sliced Tube) *The standard sliced tube is defined by*

$$\mathcal{S}(\kappa, r, I) \equiv \bigcup_{s\in I} \left(\{s\} \times P^s([-r, r]^{d-2} \times \{0\}) \times [-\kappa r\beta_s, \kappa r\beta_s]\right) ,$$

where $P^s = P^s_0$ and $\beta_s = |\langle P^s e_2, e_2\rangle|$.

Remark 5.2.6 *Notice that the standard sliced tube is a sliced tube with a change of scale in direction y. In fact, by the cocycle property of the linear Poincaré flow, we can deduce that*

$$\mathcal{S}(\kappa, r, [a, b]) = \mathcal{S}_{\mathcal{B}}(\beta_a\kappa, r, [a, b]),$$

where $\mathcal{B} = P^a([-1, 1]^{d-2} \times \{0\})$.

It is convenient to work with sliced tubes (instead of the usual tubular neighborhood) because these sets are saturated by a family of cross sections (the slices), each one being orthogonal to the base-orbit of the model vector field. The fact that the slices of the tube are intrinsically related to the linear Poincaré flow is also an important feature of these sets, since it makes possible the comparison between tubular neighborhoods and sliced tubes.

Proposition 5.2.7 (Approximation by sliced tubes) *Given $\Delta > 0$, $T > 2\Delta$, $C > 0$, $\lambda > 1$ and $K > 1$, there exist $\kappa_0 \in (0,1)$ and $\rho_0 > 0$ such that, for all (C,T,Δ)-model vector field X, for any $\kappa \in (0,\kappa_0)$, $\rho \in (0,\rho_0)$, for all interval $I = [a,b] \subset [0,T]$ satisfying $|I| > 2\Delta$ and for any K-ball $\mathcal{B} \subset \mathbb{R}^{d-2}$ we have*

$$\mathcal{S}_{\mathcal{B}}(\kappa, \rho\lambda^{-1}, [a, b-\Delta]) \subset \bigcup_{t\in[0,b-a]} \varphi^t(U_a) \subset \mathcal{S}_{\mathcal{B}}(\kappa, \rho\lambda, [a, b+\Delta]),$$

where $U_a = \{a\} \times \rho\mathcal{B} \times [-\kappa\rho, \kappa\rho]$.

Note that as $\Delta > 0$ approaches zero and as $\lambda > 1$ approaches 1, the better the approximation becomes. On the other hand, in order to achieve a good approximation of a tubular neighborhood by sliced tubes, we need to impose that the initial slice (U_a) is sufficiently small ($\rho \ll 1$) and sufficiently thin ($\kappa \ll 1$). Before proving Proposition 5.2.7, we state and prove a technical lemma which shows the relation between κ and the distortion of the sliced tube in the w-direction. This technical lemma is an adaptation of [AB1, Lemma 5] to our context.

Lemma 5.2.8 (Approximation of the linear part) *Given $\lambda > 1$, $T > 0$, $C > 0$ and $K > 1$ there exists $\kappa_0 \in (0,1)$ such that for any (C,T)-model vector field X, any $\kappa \in (0,\kappa_0)$ and for any K-ball $\mathcal{B} \subset \mathbb{R}^{d-2}$ we have*

$$(\lambda^{-1}P_s^t(\mathcal{B}\times\{0\}))\times[-\beta_t\kappa,\beta_t\kappa] \subset P_s^t(\mathcal{B}\times[-\kappa,\kappa]) \subset (\lambda P_s^t(\mathcal{B}\times\{0\}))\times[-\beta_t\kappa,\beta_t\kappa],$$

for all $[s,t] \in [0,T]$.

Proof: In order to simplify notation, we will write $P_s^t(\mathcal{B})$, instead of $P_s^t(\mathcal{B} \times \{0\})$.

By the invariance of the plane $\{y = 0\}$, we have, for all $s, t \in [0,T]$,

$$P_s^t(w,y) = \begin{pmatrix} A_s^t & B_s^t \\ 0 & \beta_s^t \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix}.$$

And in this case,

$$(P_s^t)^{-1} = P_t^s = \begin{pmatrix} (A_s^t)^{-1} & -(\beta_s^t)^{-1}(A_s^t)^{-1}B_s^t \\ 0 & (\beta_s^t)^{-1} \end{pmatrix}$$

Take

$$\kappa_0 = \frac{\lambda - 1}{K\lambda M_T},$$

where $M_T = e^{2CT}$. Also, observe that, by the sub-exponential growth of the linear flow over the base line of a model vector field,

$$\begin{aligned}\|(A_s^t)^{-1}B_s^t\| &\le \|(A_s^t)^{-1}\| \cdot \|B_s^t\| \\ &\le \|P_t^s\| \cdot \|P_s^t\| < e^{2C|t-s|} < e^{2CT}.\end{aligned}$$

Consider $0 < \kappa < \kappa_0$. First let us prove that

$$P_s^t(\mathcal{B} \times [-\kappa, \kappa]) \subset \lambda P_s^t(\mathcal{B}) \times [-\beta_s^t\kappa, \beta_s^t\kappa].$$

Assume that $w \in \mathcal{B}$ and $|y| \le \kappa$. We need only to prove that

$$A_s^t w + B_s^t y \in \lambda P_s^t(\mathcal{B}).$$

Or, equivalently, that

$$\|(A_s^t)^{-1}(A_s^t w + B_s^t y)\|_{\mathcal{B}} < \lambda,$$

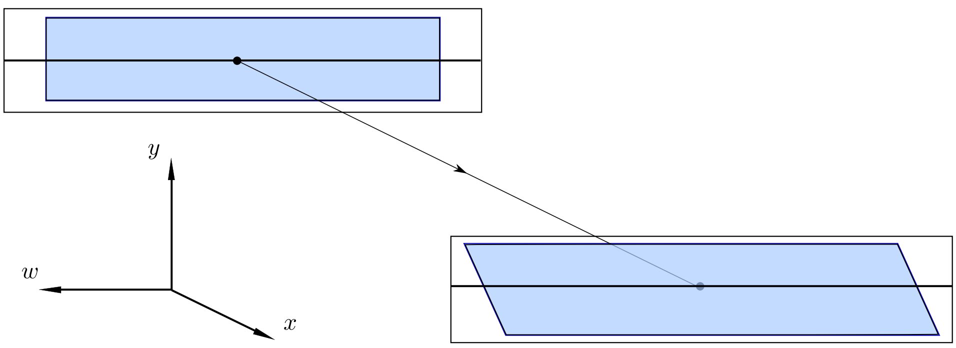

Figure 5.3: Distortion in the w-direction is not significant if the initial slice is thin enough.

Where $\|\cdot\|_{\mathcal{B}}$ is the norm given by Remark 5.2.2. Indeed,

$$\begin{aligned}\|(A_s^t)^{-1}(A_s^t w + B_s^t y)\|_{\mathcal{B}} &= \|w + (A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} \\ &\leq \|w\|_{\mathcal{B}} + \|(A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} \\ &\leq 1 + K\|(A_s^t)^{-1}B_s^t y\| \\ &\leq 1 + K\|(A_s^t)^{-1}B_s^t\| \cdot |y| \\ &\leq 1 + K \cdot M_T \kappa < \lambda,\end{aligned}$$

as we wanted to show. On the other hand, assume that

$$(\tilde{w}, \tilde{y}) \in \lambda^{-1}P_s^t(\mathcal{B}) \times [-\beta_s^t\kappa, \beta_s^t\kappa],$$

Let y be such that $|y| \leq \kappa$ and $w \in \mathcal{B}$ such that $\tilde{y} = \beta_s^t y$ and $\tilde{w} = \lambda^{-1}A_s^t w$. Then we have that

$$(P_s^t)^{-1}(\tilde{w}, \tilde{y}) = (\lambda^{-1}w - (A_s^t)^{-1}B_s^t y, y),$$

So we need only to prove that

$$\|\lambda^{-1}w - (A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} \leq 1.$$

And indeed,

$$\begin{aligned}\|\lambda^{-1}w - (A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} &\leq \lambda^{-1}\|w\|_{\mathcal{B}} + \|(A_s^t)^{-1}B_s^t y\|_{\mathcal{B}} \\ &\leq \lambda^{-1} + K\|(A_s^t)^{-1}B_s^t y\| \\ &\leq \lambda^{-1} + K \cdot M_T\kappa \leq 1.\end{aligned}$$

■

Proof of Proposition 5.2.7:

Take any $1 < \bar{\lambda} < \lambda$ and choose $\kappa > 0$ as in Lemma 5.2.8, such that

$$\begin{aligned}\bar{\lambda}^{-1}P_a^s(\mathcal{B} \times \{0\}) \times [-\kappa\beta_a^s, \kappa\beta_a^s] &\subset P_a^s(\mathcal{B} \times [-\kappa, \kappa]) \\ &\subset \bar{\lambda}P_a^s(\mathcal{B} \times \{0\}) \times [-\kappa\beta_a^s, \kappa\beta_a^s], \quad (5.8)\end{aligned}$$

for all $[a, s] \in [0, T]$.

Let $\tau_s = \tau_{s-a,(a,0,0)}$ be the hitting-time function (Definition 2.1.2) with respect to the flow φ, with time $s-a$ and base point $(a,0,0)$.

Denote by N_s the subspace $N_{\varphi^s((a,0,0))} \subset T_{\varphi^s((a,0,0))}M$ orthogonal to $X(\varphi^s((a,0,0)))$ and let $\Phi^s : U_a \to N_s$ be the Poincaré Map with respect to the flow φ and time $s-a$, that is

$$\Phi^s(p) = \varphi^{\tau_s(p)}(p).$$

Choose $\rho > 0$ small enough so that if $p \in U_a = U_a(\kappa,\rho)$ then $|\tau_s(p) - s| < \Delta$ for all $s \in [a, T+\Delta]$, and in particular,

$$\tau_{\alpha-\Delta}(p) < \alpha < \tau_{\alpha+\Delta}(p) < \tau_{\beta-\Delta}(p) < \beta < \tau_{\beta+\Delta},$$

for all $a < \alpha < \beta < T$ with $\beta - \alpha > 2\Delta$. So

$$\bigcup_{s\in[\alpha+\Delta,\beta-\Delta]} \Phi^s(U_a) \subset \bigcup_{s\in[\alpha,\beta]} \varphi^s(U_a) \subset \bigcup_{s\in[\alpha-\Delta,\beta+\Delta]} \Phi^s(U_a). \tag{5.9}$$

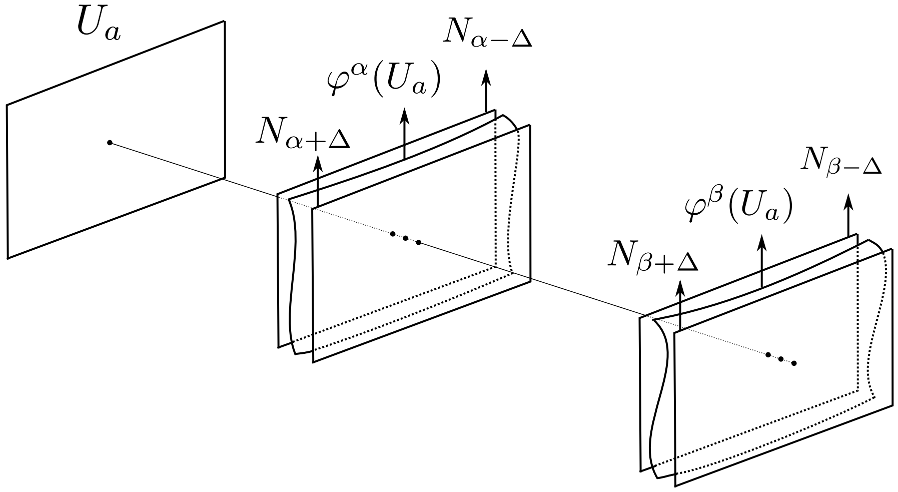

Figure 5.4: $|\tau_s(p) - s| < \Delta$ for all $s \in [a, T+\Delta]$.

Since

$$D\Phi^s_{(a,0,0)} = P^s_a \quad \text{(See Proposition 2.3.9)},$$

we can use the Taylor Formula to obtain

$$\Phi^s(p) = \underbrace{\Phi^s((a,0,0))}_{(a+s,0,0)} + P^s_a(p) + r(p)$$

where $r : U_0 \to \mathbb{R}^{d-1}$ satisfies

$$\lim_{\|p\|\to 0} \frac{\|r(p)\|}{\|p\|} = 0.$$

Notice that if r is identically zero then the conclusion of the proposition follows from (5.8) and (5.9). Taking a smaller ρ if necessary, the general case follows from Taylor approximation. ■

Corollary 5.2.9 (Standard Approximation) *Given $\Delta > 0$, $T > 2\Delta$, $C > 0$ and $\lambda > 1$, there exist $\kappa_0 \in (0,1)$ and $\rho_0 > 0$ such that, for all (C,T,Δ)-model vector field X, for any $\kappa \in (0,\kappa_0)$, $\rho \in (0,\rho_0)$ and for all interval $I = [a,b] \subset [0,T]$ satisfying $|I| > 2\Delta$ we have*

$$\mathcal{S}(\kappa, \rho\lambda^{-1}, I_\Delta) \subset \bigcup_{t\in I} \varphi^t(U_0) \subset \mathcal{S}(\kappa, \rho\lambda, I^\Delta),$$

where $U_0 = \{0\} \times [-\rho,\rho]^{d-2} \times [-\kappa\rho, \kappa\rho]$.

Proof: Recall that, by Remark 5.2.6,

$$\mathcal{S}(\kappa, \rho, [a,b]) = \mathcal{S}_{\mathcal{B}_a}(\beta_a\kappa, \rho, [a,b]),$$

where $\mathcal{B}_a = P^a([-1,1]^{d-2} \times \{0\})$. Let $\bar\kappa_0 \in (0,1)$ and $\rho_0 > 0$ be given by Proposition 5.1.1 for the K obtained by Remark 5.2.4. The Corollary follows by taking

$$\kappa_0 = \bar\kappa_0 e^{-C(T+\Delta)}$$

and by observing that $\beta_a < \|P_0^a\| < e^{C(T+\Delta)}$. ■

Remark 5.2.10 *Notice that if X is a C-model vector field with associated linear Poincaré flow given by*

$$P_s^t = \begin{pmatrix} A_s^t & B_s^t \\ 0 & \beta_s^t \end{pmatrix},$$

then the vector field given by

$$X_\epsilon(x,w,y) = X(x,w,y) - \epsilon y \cdot e_d$$

is a $(C+\epsilon)$-model vector field with the associated linear Poincaré flow given by

$$\widetilde{P}_s^t = \begin{pmatrix} A_s^t & \widetilde{B}_s^t \\ 0 & e^{-\epsilon(t-s)}\beta_s^t \end{pmatrix}.$$

Proposition 5.2.11 (Perturbed sliced tube) *Let X be model vector field, $\epsilon > 0$ and X_ϵ be as in the previous Remark, that is, $X_\epsilon(x, w, y) = X(x, w, y) - y\epsilon \cdot e_d$. Then, for any $\kappa \in (0, 1)$, $r > 0$ and any $I = [a, b] \subset [0, \infty)$, we have*

1. $\pi_x(\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I)) = \pi_x(\mathcal{S}_{\mathcal{B}}(\kappa, r, I))$;
2. $\pi_w(\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I)) = \pi_w(\mathcal{S}_{\mathcal{B}}(\kappa, r, I))$;
3. $\pi_y(\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I)) \subset \pi_y(\mathcal{S}_{\mathcal{B}}(\kappa, r, I))$,

where $\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I)$ denotes de sliced tube with respect to X_ϵ and π_x, π_w, π_y are the projections in directions x, w and y, respectively. In Particular,

$$\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, r, I) \subset \mathcal{S}_{\mathcal{B}}(\kappa, r, I).$$

Proof: The first conclusion is trivial, since the projection in the coordinate x of a sliced tube is I. The second conclusion is a direct consequence of Remark 5.2.10, since

$$\tilde{P}^t_s|_{\{y=0\}} = A^t_s = P^t_s|_{\{y=0\}}.$$

In order to see that third consequence is also true, notice that, by Remark 5.2.10,

$$\tilde{P}^t_s|_{\{w=0\}} = e^{-\epsilon(t-s)}\beta^t_s < \beta^t_s,$$

since we are considering $0 < s < t$. ■

5.3 Bump Function

In the previous subsection we saw that, in terms of volume, it is possible to approximate a tubular neighborhood by two sliced tubes: one that contains the neighborhood, and other that is contained in it. In this subsection, we show how to construct a bump function with small C^1 norm, supported on the inner (standard) sliced tube.

Remark 5.3.1 *Given a C-model vector field X, the function $\beta_t(x) = |\langle P^t_x(e_2), e_2\rangle|$, defined inside a sliced tube with respect to X is a linear cocycle in $\mathbb{R}$ over φ^t_X. Moreover, its logarithmic derivative is bounded by C. In other words, if we fix $x \in \mathbb{R}$ and let t vary, we have that*

$$\left|\frac{\beta_t(x)'}{\beta_t(x)}\right| \leq C.$$

Proposition 5.3.2 (Bump Function) *Given $\epsilon > 0$ and $0 < \gamma < 1$ there exists $0 < \epsilon' < \epsilon$ such that for all $T_0, a, C > 0$ there exist $0 < \kappa_0 < 1$ and*

$r_0 > 0$ such that for all (C, T_0, a)-model vector field X and for any $T \in (2a, T_0)$, $\kappa \in (0, \kappa_0)$ and $r \in (0, r_0)$ there exists a function $\Psi : \mathbb{R}^d \to \mathbb{R}$ such that

1. *$\Psi = 0$ outside $\mathcal{S}(\kappa, r, [-a, T + a])$;*
2. *$\Psi = \epsilon'$ on $\mathcal{S}(\kappa, r(1 - \gamma), [a, T - a])$;*
3. *$\|y\Psi(x, w, y)\|_{C^1} < \epsilon$, for all $(x, w, y) \in \mathbb{R}^d$.*

Proof: Let $\epsilon' = \dfrac{\epsilon\gamma}{2 + \gamma}$ and fix any $C > 0$, any large $T_0 > 0$ and any small $a > 0$. Define

$$\kappa_0 = e^{-2C(T_0+2a)}$$

and take $T \in (2a, T_0)$, $\kappa \in (0, \kappa_0)$ and any (C, T_0, a)-model vector field X. Observe that, by the third property of model vector fields, by Remarks 2.6.1 and 5.3.1, we can conclude that

$$\begin{aligned}(\beta_x \|(P^x)^{-1}\|)^{-1} &< \sup \beta_x^{-1} \cdot \sup \|(P^x)^{-1}\|^{-1} \\ &= (\inf \beta_x)^{-1} \cdot (\inf \|(P^x)^{-1}\|)^{-1} \\ &< e^{-2Cx} < e^{-2C(T_0+2a)} = \kappa_0.\end{aligned}$$

Consider a bump-function $\xi_1 : \mathbb{R} \to \mathbb{R}$ such that:

- $\xi_1(x) = 0$, if $x \in \mathbb{R}\backslash[-1, 1]$;
- $\xi_1(x) = 1$, if $x \in [-1 + \gamma, 1 - \gamma]$;
- $|\xi_1'(x)| \leq \frac{2}{\gamma}$, for all $x \in \mathbb{R}$.

Also define another bump-function $\xi_2 : \mathbb{R} \to \mathbb{R}$ such that

- $\xi_2(x) = 0$, if $x \in \mathbb{R}\backslash[-a, T + a]$;
- $\xi_2(x) = 1$, if $x \in [a, T - a]$,
- $|\xi_2'(x)| \leq \frac{2}{a}$, for all $x \in \mathbb{R}$.

For some $r > 0$ (we will estimate r a posteriori) we define the bump-function

$$\begin{array}{rccl}\Psi : & \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{d-2} & \to & \mathbb{R} \\ & (x, w, y) & \mapsto & \epsilon'\xi_1(\kappa^{-1}r^{-1}\beta_x^{-1}y)\xi_1(r^{-1}\|(P^x)^{-1}w\|)\xi_2(x)\end{array}$$

Properties 1 and 2 are very easy to check and we only need to estimate the derivative of $y\Psi(x,w,y)$ inside the tube $\mathcal{S}(\kappa, r, [-a, T+a])$. So in next computations we can always assume that

$$|y| < \kappa\beta_x r \tag{5.10}$$

and also that

$$\|(P^x)^{-1}w\| < r. \tag{5.11}$$

In order to prove that

$$\|y\Psi(x,w,y)\|_{C^1} < \epsilon,$$

we first observe that

$$|y\Psi(x,w,y)| < |y|\epsilon' < \epsilon.$$

Now let us compute the derivatives.

1.

$$\begin{aligned}
\left|\frac{\partial}{\partial y}(y\Psi(x,w,y))\right| &< |\Psi(x,w,y)| + |y|\left|\frac{\partial}{\partial y}(\Psi(x,w,y))\right| \\
&< \epsilon' + |y|\left(\frac{\epsilon'}{\kappa r\beta_x}|\xi_1'(\kappa^{-1}r^{-1}\beta_x^{-1}y)| \cdot |\xi_1(r^{-1}\|(P^x)^{-1}w\|)| \cdot |\xi_2(x)|\right) \\
&< \epsilon'\left(1 + \frac{2}{\gamma}\right) \\
&< \frac{\epsilon\gamma}{(\gamma+2)}\frac{(\gamma+2)}{\gamma} = \epsilon;
\end{aligned}$$

2.

$$\begin{aligned}
\|D_3(y\Psi(x,w,y))\| &< \epsilon'\frac{|y|}{r}|\xi_1'(r^{-1}\|(P^x)^{-1}w\|)| \cdot \|(P^x)^{-1}\| \\
&< \epsilon'\frac{\kappa r\beta_x}{r}\frac{2}{\gamma}\|(P^x)^{-1}\| \\
&< \epsilon'\frac{2}{\gamma}\kappa\beta_x\|(P^x)^{-1}\|,
\end{aligned}$$

By the choice of κ_0,

$$\kappa\beta_x\|(P^x)^{-1}\| < 1$$

and we have:

$$\begin{aligned}\|D_3(y\Psi(x,w,y))\| &< \epsilon'\frac{2}{\gamma} \\ &< \frac{\epsilon\gamma}{2+\gamma}\cdot\frac{2}{\gamma} \\ &< \frac{\epsilon}{1+\frac{\gamma}{2}} < \epsilon;\end{aligned}$$

3.

$$\begin{aligned}\left|\frac{\partial}{\partial x}(y\Psi(x,w,y))\right| &< \epsilon'|y|\cdot\left|\frac{\partial}{\partial x}\left(\xi_1(\kappa^{-1}r^{-1}\beta_x^{-1}y)\xi_1(r^{-1}\|(P^x)^{-1}w\|)\xi_2(x)\right)\right| \\ &< \epsilon'|y|\left|\left[\xi_1\left(\frac{y}{\kappa r\beta_x}\right)\right]'\right| + \left|\left[\xi_1\left(\frac{\|(P^x)^{-1}w\|}{r}\right)\right]'\right| + |\xi_2'(x)| \\ &< 2\epsilon'|y|\left(\frac{|y|}{\gamma\kappa r\beta_x^2}\cdot\frac{\partial\beta_x}{\partial x} + \frac{1}{\gamma r}\cdot\frac{\partial(\|(P^x)^{-1}w\|)}{\partial x} + \frac{1}{a}\right)\end{aligned}$$

By Remark 5.3.1, we have that $\left|\frac{\beta_x'}{\beta_x}\right| < C$ for all $x \in [-a, T_0 + a]$. This estimate, together with the third property of model vector fields, gives us

$$\left|\frac{\partial}{\partial x}(y\Psi(x,w,y))\right| < 2\epsilon'\left(\frac{\kappa r\beta_x C}{\gamma} + \frac{\kappa r\beta_x C}{\gamma} + \frac{\kappa r\beta_x}{a}\right).$$

We can assume that $r > 0$ was chosen sufficiently small in order to obtain

$$\begin{aligned}\left|\frac{\partial}{\partial x}(y\Psi(x,w,y))\right| &< \frac{2\epsilon'}{\gamma} \\ &< \frac{\epsilon\gamma}{2+\gamma}\cdot\frac{2}{\gamma} \\ &< \frac{\epsilon}{1+\frac{\gamma}{2}} < \epsilon;\end{aligned}$$

Observe that the constant $r > 0$ did not influence the estimates of the derivative in the y and w directions and this scale invariance is expected when we work with C^1-perturbations. The fact that $r > 0$ is being used in the estimate of the derivative along the x-direction may seem strange, but notice that r is not related with the length (x-direction) of the sliced tube, but only with its transverse size. Indeed, the greater the length of the tube in relation to its thickness, less restrictive is the bump-function's derivative along the x-direction. ∎

5.4
Proof of the Fettuccine's Lemma

We will verify the conclusion of the Lemma for the pulled-back vector field F_*X and define a δ-crushable set U in $\mathbb{R}^d$ with respect to the C-sliced measure $\hat{m}$. In order to see that this Euclidean version of the Lemma is sufficient for the conclusion of the proof, notice that, since the perturbation is given by adding vertical vectors to the pulled-back vector field, Proposition 4.0.19 guarantees that the pushed-forward of the perturbation will be a $C\epsilon$-perturbation of the original vector field (recall that $C > 1$ depends only on the vector field $X \in \mathfrak{X}^3(M)$ and the cross section Σ). Moreover, since the C-sliced measure $\hat{m}$ is comparable to $F_*^{-1}(m)$, the relative δ-crushing property of $F_*(X)$ on U will originate a 4δ-crushing property in $F(U)$.

Let us fix $\epsilon > 0$ and $0 < \delta < 1$. Once defined the auxiliary constants

$$\gamma = 1 - \sqrt[d]{1 - \delta/2} \tag{5.12}$$

and

$$\delta' = (1 - \gamma)\delta, \tag{5.13}$$

we can choose $\epsilon' = \epsilon'(\epsilon, \gamma)$ from Proposition 5.3.2 (Bump Function) and $t_0 = t_0(\epsilon', \delta') > 1$, the crushing-time from Lemma 5.1.2. Now take any $T_0 > 3t_0$ and assume that $F : Z \to F(Z)$ is the tubular chart, given by Theorem 4.0.15 with respect to a non-periodic point $p \in \Sigma$ and time T_0. Notice that the sliced measure $\hat{m}$ with density ω of a standard sliced tube is given by

$$\hat{m}(\mathcal{S}(\kappa, r, I)) = 2\kappa r^{d-1} \int_I f(s)ds,$$

where

$$f(s) = |\det(P^s)| \cdot \omega(s)$$

and $P^s = P_0^s$ is the family of linear isomorphisms induced by the linear Poincaré flow. Observe that f has bounded logarithmic derivative (See §2.6), so we can use Proposition 2.6.2 to find a constant $a_0 = a_0(t_0, \gamma) \leq 1$ such that, for all interval $I \subset [0, T_0]$ with $|I| \geq t_0$ and for all $0 < a < a_0$,

$$\int_{I_a} f(s)ds > (1-\gamma)\int_{I^a} f(s)ds. \tag{5.14}$$

Recall that Property 10 of the tubular chart (Theorem 4.0.15) ensures that the bound of the logarithmic derivative of $f(s)$ depends only on the original vector field $X \in \mathfrak{X}^3(M)$. In fact, every time we evoke results of this Section about model vector fields, we will be implicitly using this Property.

Let $\kappa_0 = \kappa_0(T_0, a_0)$ and $\rho_0 = \rho_0(T_0, a_0)$ be as in Proposition 5.3.2 (Bump Function) and take $\lambda > 1$ such that

$$\lambda < \sqrt[3(d-1)]{\frac{1-\frac{\delta}{2}}{1-\delta}}. \tag{5.15}$$

With these choices of T_0, a_0 and λ, we can find $\kappa_1 > 0$ and $\rho_1 > 0$ from Corollary 5.2.9 (Standard approximation) in order to obtain, for any $I \subset [0, T_0]$, with $|I| \geq t_0$, for any $a \in (0, a_0)$, $\kappa \in (0, \kappa_1)$ and $\rho \in (0, \rho_1)$,

$$\mathcal{S}(\kappa, \rho\lambda^{-1}, I_a) \subset \bigcup_{s\in I} \varphi^s_X(U_0) \subset \mathcal{S}(\kappa, \rho\lambda, I^a), \tag{5.16}$$

where $U_0 = \{0\} \times [-\kappa\rho, \kappa\rho] \times [\rho, \rho]^{d-2}$. Let $K > 1$ be the constant given by Remark 5.2.4 and recall, that, by Remark 5.2.10, the vector field

$$X_{\epsilon'}(x, w, y) = X(x, w, y) - \epsilon' y \cdot e_d$$

is a $(C+\epsilon')$-model vector field. Thus, we can reduce κ_1 and ρ_1, if necessary, and use Proposition 5.2.7 to obtain, for any K-ball $\mathcal{B}$, any $I \subset [s, T]$ with $|I| \geq t_0$, any $a \in (0, a_0)$, $\kappa \in (0, \kappa_1)$ and $\rho \in (0, \rho_1)$,

$$\widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, \rho\lambda^{-1}, [s, T-a]) \subset \bigcup_{t\in[0,T-s]} \varphi^t_{\widetilde{X}}(U_s) \subset \widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa, \rho\lambda, [s, T+a]), \tag{5.17}$$

where $U_s = \{s\} \times (\rho\mathcal{B} \times \{0\}) \times [\kappa\rho, \kappa\rho]$.

Take

$$\kappa = e^{-C(T_0+a_0)} \min\{\kappa_0, \kappa_1\}.$$

Now, choose any $3t_0 < T < T_0$ and define $\rho_2 = \rho_2(T, a_0, \gamma)$ such that Lemma 5.1.2 (Crushing time) is satisfied. and $\rho = \min\{\rho_0, \rho_1, \rho_2\}$ and define a bump-function $\Psi = \Psi_{\lambda^{-1}\rho,\kappa}$ given by Proposition 5.3.2 such that

1. $\Psi \equiv 0$ outside $\mathcal{S}(\kappa, \lambda^{-1}\rho, [a, T-a])$;
2. $\Psi \equiv \epsilon'$ in $\mathcal{S}(\kappa, \lambda^{-1}(1-\gamma)\rho, [2a, T-2a])$;
3. $\|y\Psi\|_{C^1} < \epsilon$.

We define the perturbed vector field in $\mathbb{R}^d$ by

$$\widetilde{X}(x,w,y) \mapsto X(x,w,y) - y\Psi(x,w,y)$$

and denote the set to be crushed by

$$V = \mathcal{S}(\kappa, \rho\lambda^{-2}(1-\gamma), [2a, T - t_0 - 3a]).$$

Now let us verify that our choices were appropriate. First, note that, by relation (5.16) and by the construction of Ψ, $\widetilde{X}$ is indeed an ϵ-perturbation of X with support on

$$\mathcal{S}(\kappa, \rho\lambda^{-1}, [a, T-a]) \subset \bigcup_{t\in[0,T]} \varphi^t(U_0) = U.$$

Let us introduce some auxiliary notation:

$$\begin{aligned} S^- &= \mathcal{S}(\kappa, \rho\lambda, [-a, T - t_0 + a]) \\ S^+ &= \mathcal{S}(\kappa, \rho\lambda^{-1}(1-\gamma), [t_0 + a, T - 2a]). \end{aligned}$$

Relation (5.16) ensures that $V \subset U^-$ and also that $U^- \subset S^-$. Therefore, we have that

$$\frac{\hat{m}(V)}{\hat{m}(U^-)} > \frac{\hat{m}(V)}{\hat{m}(S^-)} = \frac{2\kappa(\rho(1-\gamma)\lambda^{-2})^{d-1}\int_{2a}^{T-t_0-3a} f(s)ds}{2\kappa(\rho\lambda)^{d-1}\int_{-a}^{T-t_0+a} f(s)ds}.$$

The above Inequality, together with (5.12) , (5.14) and (5.15), leads us to

$$\begin{aligned} \frac{\hat{m}(V)}{\hat{m}(U^-)} &> \left(\frac{1-\gamma}{\lambda^3}\right)^{d-1} \cdot \frac{\int_{2a}^{T-t_0-3a} f(s)ds}{\int_{-a}^{T-t_0+a} f(s)ds} \\ &> \frac{(1-\gamma)^{d-1}}{\lambda^{3(d-1)}} \cdot \frac{\int_{2a}^{T-t_0-2a} f(s)ds}{\int_{-2a}^{T-t_0+2a} f(s)ds} \\ &> \frac{(1-\gamma)^d}{\lambda^{3(d-1)}} \\ &> \frac{1-\delta/2}{\frac{1-\delta/2}{1-\delta}} = 1 - \delta. \end{aligned}$$

Claim 5.4.1

$$\bigcup_{t\in[0,t_0]} \varphi^t_{\widetilde{X}}(V) \in \mathcal{S}(\kappa, \rho\lambda^{-1}(1-\gamma), [a, T-2a]).$$

Proof of the Claim: Let $p \in V$. Then there exists $s \in [2a, T-t_0-3a]$ such that

$$p \in U_s = \{s\} \times P^s([-\rho', \rho']^{d-2} \times \{0\}) \times [-\kappa\rho'\beta_s, \kappa\rho'\beta_s],$$

where $\rho' = \rho(1-\gamma)\lambda^{-2}$. By the sub-exponential growth of β_s and the choice of κ,

$$\kappa\beta_s < \kappa e^{C(T+a)} < \kappa_1$$

and we can apply Relation (5.17) to deduce, together with Remark 5.2.10, that

$$\bigcup_{t\in[0,t_0]} \varphi^t_{\widetilde{X}}(U_s) \subset \widetilde{\mathcal{S}}_{\mathcal{B}}(\kappa\beta_s, \lambda\rho', [s, s+t_0+a]) \subset \mathcal{S}_{\mathcal{B}}(\kappa\beta_s, \lambda\rho', [s, s+t_0+a]),$$

where $\mathcal{B} = P^s([-\rho', \rho'])$. By Remark 5.2.6,

$$\bigcup_{t\in[0,t_0]} \varphi^t_{\widetilde{X}}(p) \in \mathcal{S}(\kappa, \lambda\rho', [s, s+t_0+a]).$$

Since $2a < s < T - t_0 - 3_a$,

$$\varphi^t_{\widetilde{X}}(p) \in \mathcal{S}(\kappa, \rho(1-\gamma)\lambda^{-1}, [2a, T-2a])$$

and the Claim is proved. ■

The above claim we just proved together with relation (5.16) implies Item 3 of the Main Lemma. So we need only to prove item 4, that is,

$$\frac{\hat{m}(\varphi^{t_0}_{\widetilde{X}}(V))}{\hat{m}(U^+)} < \delta.$$

For that matter, let us define the set

$$W = \mathcal{S}(\delta'\kappa, (1-\gamma)\rho\lambda^{-1}, [t_0, T-2a]).$$

Since

$$\frac{\hat{m}(W)}{\hat{m}(S^+)} = \frac{\delta' \int_{t_0}^{T-2a} f(s)ds}{\int_{t_0+a}^{T-2a} f(s)ds}$$

$$< \frac{\delta' \int_{t_0-a}^{T} f(s)ds}{\int_{t_0+a}^{T-2a} f(s)ds}$$

$$< \frac{\delta'}{(1-\gamma)} = \delta$$

and since $S^+ \subset U^+$, for finishing the proof of the Lemma, it is sufficient to show that $\varphi_{\widetilde{X}}^{t_0}(V) \subset W$.

Note that Claim 5.4.1 placed us in the context of Lemma 5.1.2 (Crushing time) because

$$\psi = \epsilon' \quad \text{in} \quad \mathcal{S}(\kappa, \rho\lambda^{-1}(1-\gamma), [a, T-2a])$$

and, consequently, in this set, the perturbation $\widetilde{X}$ is the same as in Lemma 5.1.2.

Assume that $p_2 = (s, y_2, w_2) \in \varphi_{\widetilde{X}}^{t_0}(V)$ and let $p_1 = (r, y_1, w_1) \in V$ such that $\varphi_{\widetilde{X}}^{t_0}(p_1) = p_2$.

Let $\widetilde{\Phi} : N_{s-t_0} \to N_s$ be the Poincaré map with respect to the perturbed flow and $q \in N_{s-t_0}$ be such that $\widetilde{\Phi}(q) = p_2$, that is,

$$\varphi_{\widetilde{X}}^{\tilde{\tau}(q)}(q) = p_2.$$

Without loss of generality, we can assume that $\rho > 0$ is sufficiently small to guarantee that

$$|s - r - t_0| < a$$

and so that

$$t_0 < s < T - 2a. \tag{5.18}$$

By Claim 5.4.1, we have that

$$w_2 \in P^s([-\rho(1-\gamma)\lambda^{-1}, \rho(1-\gamma)\lambda^{-1}]^{d-2}) \tag{5.19}$$

and Lemma 5.1.2 implies that

$$\begin{aligned} |y_2| &= |\pi(\varphi_{\widetilde{X}}^{\tilde{\tau}(q)}(q))| \\ &< \delta'|\pi(\varphi^{\tau(q)}(q))| \\ &< \delta'\rho\kappa(1-\gamma)\lambda^{-1}\beta_s. \end{aligned}$$

The above Inequality together with (5.18) and (5.19) implies that $p_2 \in W$.